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LETTER TO THE EDITOR

Characters of Hecke algebras $H_n(q)$ of type A_{n-1}

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Abstract. The connection between the Ocneanu trace on $H_n(q)$ and Schur functions leads to a simple method for calculating the irreducible characters of the Hecke algebras $H_n(q)$. The characters appear as the elements of the transition matrix relating certain generalized power sum symmetric functions to Schur functions.

The Hecke algebra representations of braid groups and link polynomials has been reviewed by Jones (1987). The traces of the Hecke algebras $H_n(q)$ of type A_{n-1} play a key role in applications. The traces may be calculated from the irreducible representations of $H_n(q)$ as constructed by a number of workers (Dipper and James 1987 and Wenzl 1989). It would be desirable to be able to determine the traces directly, in much the same way as occurs for the symmetric group S_n (Littlewood 1940), without first constructing explicit representations. We present such a method in this letter.

The complex Hecke algebra $H_n(q)$, with q an arbitrary but fixed complex parameter, is generated by g_i with $i = 1, 2, \ldots, n-1$ subject to the relations:

$$g_i^2 = (q-1)g_i + q \qquad \text{for } i = 1, 2, \ldots, n-1 \tag{1}$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \qquad \text{for } i = 1, 2, \ldots, n-2 \tag{2}$$

$$g_i g_j = g_j g_i \qquad \text{for } |i-j| \geqslant 2. \tag{3}$$

For $q = 1$ these relations are exactly those appropriate to the symmetric group S_n with g_i replaced by s_i for $i = 1, 2, \ldots, n-1$, where s_i is the transposition $(i, i+1)$. Every permutation π in S_n can be expressed as a reduced word of minimal length $l(\pi)$ in the generators s_i. There exists a map h from S_n to $H_n(q)$ such that $h(s_i) = g_i$ and $h(\pi) = g_{i_1} g_{i_2} \ldots g_{i_m}$ for any permutation $\pi = s_{i_1} s_{i_2} \ldots s_{i_m} \in S_n$. The set of reduced words $h(\pi)$ for all $n!$ permutations $\pi \in S_n$ forms a basis of $H_n(q)$.

Ocneanu (Ocneanu 1985†, Freyd *et al* 1985) has defined a linear trace on $H_\infty(q)$, the inductive limit of $H_n(q)$ as $n \to \infty$, such that for each $z \in \mathbb{C}$

$$\mathrm{tr}(xy) = \mathrm{tr}(yx) \tag{4}$$

$$\mathrm{tr}(1) = 1 \tag{5}$$

$$\mathrm{tr}(xg_n) = z\,\mathrm{tr}(x) \qquad \text{for } x \in H_{n-1}(q). \tag{6}$$

The defining relations (1)–(3) augmented by (4) may be used to express the trace of a given element x of $H_\infty(q)$ as a linear sum of traces of certain minimal words which are both reduced and contain no generator g_i more than once. It follows from (5) and (6) that the trace of each minimal word, v, is given by

$$\mathrm{tr}(v) = z^{l(v)}. \tag{7}$$

† A cryptic summary appears in Freyd *et al* (1985) with some details being given in Jones (1987) and Wenzl (1989).

Each minimal word v consists of a unique sequence of subwords $v_1, v_2, \ldots$ which we call connected words, where a word is said to be connected if it is a sequence of consecutive generators $g_i g_{i+1} g_{i+2} \ldots$. There are 2^{n-1} distinct minimal words in $H_n(q)$ but they can conveniently be assigned to classes labelled by distinct partitions ρ of n in accordance with their connectivity. A minimal word v of $H_n(q)$ is said to have connectivity $\rho = (\rho_1, \rho_2, \ldots)$ if it takes the form:

$$v = (g_{i_1} g_{i_1+1} \ldots g_{i_1+r_1-2})(g_{i_2} g_{i_2+1} \ldots g_{i_2+r_2-2}) \ldots (g_{i_k} g_{i_k+1} \ldots g_{i_k+r_k-2}) \tag{8}$$

with

$$i_1 + r_1 - 1 < i_2 \qquad i_2 + r_2 - 1 < i_3 \quad \ldots \quad i_{k-1} + r_{k-1} - 1 < i_k$$

where $(r_1, r_2, \ldots, r_k)$ is a permutation of those parts of $\rho > 1$.

Specifying the connectivity classes of minimal words of $H_n(q)$ by partitions of n is in accord with the practice of using such partitions to specify conjugacy classes of the elements of S_n as dictated by their cycle structure. Indeed the minimal word $v = h(\pi)$ in $H_n(q)$ belongs to the connectivity class labelled by ρ if and only if its pre-image π in S_n belongs to the conjugacy class labelled by ρ.

The characters of the symmetric group S_n are the elements of a transition matrix that relates the power sum symmetric functions $p_\rho(\boldsymbol{t})$ to the Schur functions $s_\lambda(\boldsymbol{t})$ for an arbitrary set of indeterminates $\boldsymbol{t} = (t_1, t_2, \ldots)$ (Macdonald 1979). We may generalize the power sum functions by letting

$$p_r(q; \boldsymbol{t}) = \sum_{\substack{a,b=0 \\ a+b+1=r}}^{r-1} (-1)^b q^a s_{(a+1,1^b)}(\boldsymbol{t}) \tag{9}$$

and for $\rho = (\rho_1, \rho_2, \ldots)$ letting

$$p_\rho(q; \boldsymbol{t}) = p_{\rho_1}(q; \boldsymbol{t}) p_{\rho_2}(q; \boldsymbol{t}) \ldots . \tag{10}$$

This then enables us to state the following key theorem:

Theorem 1. Let v be any minimal word of $H_n(q)$ having connectivity ρ. Then the character of v in the irreducible representation π_λ is given by

$$\operatorname{tr} \pi_\lambda(v) = \chi^\lambda_\rho(q) \tag{11}$$

where $\chi^\lambda_\rho(q)$ is defined by the generating function

$$p_\rho(q; \boldsymbol{t}) = \sum_\lambda \chi^\lambda_\rho(q) s_\lambda(\boldsymbol{t}). \tag{12}$$

The proof of theorem 1 depends upon the validity of a remarkable formula (Jones 1987, Reshetikhin 1988) relating the Ocneanu trace on the Hecke algebra $H_n(q)$ to the characters of irreducible representations. Provided that q is not a root of unity:

$$\operatorname{tr}(x) = \sum_\lambda W_\lambda(q, z) \operatorname{tr} \pi_\lambda(x) \tag{13}$$

for all $x \in H_n(q)$, where the summation is over all partitions λ of n and

$$W_\lambda(z, z) = \prod_{(i,j) \in F^\lambda} \frac{(wq^{i-1} - zq^{j-1})}{(1 - q_{h_{ij}})} \tag{14}$$

where $w = 1 - q + z$. F^λ is the Young diagram specified by λ and h_{ij} is the hook-length of the box in the ith row and jth column of F^λ.

In fact $W_\lambda(q, z)$ is nothing other than the particular specialization of the Schur function $s_\lambda(\boldsymbol{t})$ given by (Littlewood 1940 p 125):

$$W_\lambda(q, z) = s_\lambda(w\boldsymbol{q}/z\boldsymbol{q}) \tag{15}$$

where $\boldsymbol{q} = (1, q, q^2, \ldots)$ and $p\boldsymbol{q} = (p, pq, pq^2, \ldots)$ for all p. The symbol / in the argument of the Schur function indicates the separation of the indeterminates into two sets appropriate to what are known as supersymmetric Schur functions (King 1983).

In the case of a minimal word v we obtain

$$z^{l(v)} = \sum_{\lambda} s_\lambda(w\boldsymbol{q}/z\boldsymbol{q}) \operatorname{tr} \pi_\lambda(v). \tag{16}$$

Comparison between (16) and (12) allows us to prove theorem 1 from:

Lemma 1. In the case $\boldsymbol{t} = (w\boldsymbol{q}/z\boldsymbol{q})$ with $w = 1 - q + z$

$$p_r(q; \boldsymbol{t}) = p_r(q; w\boldsymbol{q}/z\boldsymbol{q}) = z^{r-1}. \tag{17}$$

This lemma may itself be proved using the properties of ordinary and supersymmetric Schur functions and the further lemma:

Lemma 2. In the case $\boldsymbol{t} = (q/1)$

$$s_\lambda(\boldsymbol{t}) = s_\lambda(q/1) = \begin{cases} (q-1)q^a(-1)^b & \text{if } \lambda = (a+1, 1^b) \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

It is now a simple task to evaluate the characters $\chi^\lambda_\rho(q)$ by using (9), (10) and (12). The procedure involves the repeated use of the Littlewood–Richardson rule (Littlewood 1940 p 94, Macdonald 1979 p 62) for decomposing products of Schur functions:

$$s_\mu(\boldsymbol{t}) s_\nu(\boldsymbol{t}) = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda(\boldsymbol{t}). \tag{19}$$

As an example of such calculations, consider the case of (12) for which $\rho = (432)$. It follows from (9) and (10) that

$$p_{(432)}(q) = = (q^2 s_{31} + q s_{21^2} - s_{1^4})(q^2 s_3 - q s_{21} + s_{1^3})(q s_2 - s_{1^2}). \tag{20}$$

Using the Littlewood–Richardson rule to multiply out the Schur-function products we obtain

$$\begin{aligned} s_9 q^6 &+ s_{81}(2q^6 - 3q^5) + s_{72}(3q^6 - 5q^5 + 3q^4) + s_{71^2}(q^6 - 6q^5 + 5q^4) \\ &+ s_{63}(3q^6 - 6q^5 + 4q^4 - q^3) + s_{621}(2q^6 - 10q^5 + 13q^4 - 6q^3) \\ &+ s_{61^3}(-3q^5 + 10q^4 - 6q^3) + s_{54}(2q^6 - 4q^5 + 3q^4 - q^3) \\ &+ s_{531}(2q^6 - 10q^5 + 14q^4 - 8q^3 + 2q^2) + s_{52^2}(q^6 - 4q^5 + 9q^4 - 9q^3 + 3q^2) \\ &+ s_{521^2}(-5q^5 + 17q^4 - 18q^3 + 7q^2) + s_{51^4}(5q^4 - 12q^3 + 5q^2) \\ &+ s_{4^21}(q^6 - 4q^5 + 6q^4 - 4q^3 + q^2) + s_{432}(q^6 - 4q^5 + 9q^4 - 9q^3 + 5q^2 - q) \\ &+ s_{431^2}(-4q^5 + 13q^4 - 16q^3 + 8q^2 - 2q) \\ &+ s_{42^21}(-2q^5 + 8q^4 - 16q^3 + 13q^2 - 4q) + s_{421^3}(7q^4 - 18q^3 + 17q^2 - 5q) \\ &+ s_{41^5}(-6q^3 + 10q^2 - 3q) + s_{3^3}(-q^5 + q^4 - 2q^3 + q^2 - q) \\ &+ s_{3^221}(-q^5 + 5q^4 - 9q^3 + 9q^2 - 4q + 1) + s_{3^21^3}(3q^4 - 9q^3 + 9q^2 - 4q + 1) \\ &+ s_{32^3}(q^4 - 4q^3 + 6q^2 - 4q + 1) + s_{32^21^2}(2q^4 - 8q^3 + 14q^2 - 10q + 2) \\ &+ s_{321^4}(-6q^3 + 13q^2 - 10q + 2) + s_{31^6}(5q^2 - 6q + 1) \\ &+ s_{2^41}(-q^3 + 3q^2 - 4q + 2) s_{2^31^3}(-q^3 + 4q^2 - 6q + 3) + s_{2^21^5}(3q^2 - 5q + 3) \\ &+ s_{21^7}(-3q + 2) + s_{1^9}. \end{aligned}$$

The q-polynomial coefficients of the Schur functions s_λ are the non-zero q-dependent characters χ^λ_{342} of $H_9(q)$. Putting $q=1$ yields

$$s_9 - s_{81} + s_{72} - s_{621} + s_{61^3} + s_{521^2} - 2s_{51^4} + s_{432} - s_{431^2} - s_{42^21} + s_{421^3} + s_{41^5}$$
$$- 2s_{3^3} + s_{3^221} - s_{321^4} + s_{2^21^5} - s_{21^7} + s_{1^9}$$

where the coefficients of the Schur functions s_λ are now the non-zero characters χ^λ_{432} of S_9.

There is no difficulty, in principle, in evaluating the characters for representative minimal words of any connectivity class (ρ) and hence constructing of character tables for the Hecke algebras $H_n(q)$ of type A_{n-1}. Typically we obtain for $n=3$

$$\begin{array}{c} \\ \{3\} \\ \{21\} \\ \{1^3\} \end{array} \begin{array}{c} \begin{array}{ccc} (1^3) & (21) & (3) \end{array} \\ \begin{pmatrix} 1 & q & q^2 \\ 2 & -1+q & -q \\ 1 & -1 & 1 \end{pmatrix} \end{array}$$

where the rows are labelled by $\{\lambda\}$ for each irreducible representation π_λ and the columns are labelled by the connectivity classes (ρ).

The Littlewood-Richardson coefficients determine the restriction of characters of S_{m+n} to products of characters of $S_m \oplus S_n$, and they play precisely the same role in the Hecke algebra context.

Theorem 2. Let ρ correspond to a minimal word of $H_{m+n}(q)$ with $\rho=(\sigma\tau)$, where σ and τ correspond to minimal words of the subalgebras isomorphic to $H_m(q)$ and $H_n(q)$, generated by $g_1, g_2, \ldots, g_{m-1}$ and by $g_{m+1}, g_{m+2}, \ldots, g_{m+n-1}$ respectively. Then

$$\chi^\lambda_\rho(q) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} \chi^\mu_\sigma(q) \chi^\nu_\tau(q). \tag{21}$$

The proof of theorem 2 follows from consideration of (10), (11) and (19).

There follow three useful corollaries to theorems 1 and 2.

Corollary 1. If $\rho=\sigma r$, where r signifies a one part partition, then

$$\chi^\lambda_\rho(q) = \chi^\lambda_{\sigma r}(q) = \sum_{\substack{a,b=0 \\ a+b+1=r}}^{r-1} (-1)^b q^a \chi^{\lambda/(a+1,1^b)}_\sigma(q) \tag{22}$$

where

$$\chi^{\lambda/(a+1,1^b)}_\sigma(q) = \sum_\mu c^\lambda_{\mu,(a+1,1^b)} \chi^\mu_\sigma.$$

Corollary 2. If $\rho=\sigma 1$, then

$$\chi^\lambda_\rho(q) = \chi^\lambda_{\sigma 1}(q) = \chi^{\lambda/1}_\sigma(q). \tag{23}$$

Corollary 3. If $\rho=n$, then

$$\chi^\lambda_n = \begin{cases} (-1)^b q^a & \text{if } \lambda=(a+1,1^b) \text{ with } a+b+1=n \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

We conclude by noting that theorem 2 can be generalized to prove that the Littlewood-Richardson rule applies to $H_n(q)$ just as it does to S_n. A detailed account of the result sketched in this letter, along with illustrative tables and a new construction of the irreducible representation matrices making use of q-analogues of Young operators and Garnir elements, will be published elsewhere.

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